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LETTER TO THE EDITOR

The parallel critical field of a type-II superconducting cylinder

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Abstract. The analogue of the surface nucleation field H_{c3} is calculated for a superconducting cylinder in a magnetic field parallel to its axis. For small radius, on the scale of the coherence length, superconductivity nucleates uniformly across the cylinder, while for large radius a surface sheath nucleates at the outer perimeter, the bulk of the cylinder remaining normal. The transition between these two limits is seen as a succession of flux entry points each corresponding to an increase by unity in the magnitude of the fluxoid quantum number.

The prediction by Saint-James and de Gennes (1963) that superconductivity would appear at the surface of a type-II superconductor in a parallel field $H_{c3} = 1.69 H_{c2}$, where H_{c2} is the bulk critical field, has been verified in subsequent experiments, for example that of Gygax and Kropschot (1964). The derivation is based on the Ginzburg–Landau (GL) theory, and the appearance of the surface sheath at H_{c3} is a consequence of the boundary condition that the derivative of the wave function Ψ is zero at the surface. The surface sheath is suppressed if the superconductor is plated with a normal metal because the proximity effect between the normal and superconducting metals modifies the GL boundary condition; the theory is given by Hurault (1966).

A natural extension of the Saint-James–de Gennes calculation is to a thin film in a parallel field. As was first appreciated by Sutton (1966), the nature of the superconducting state just below H_{c3} depends on the dimensionless ratio $d/\xi(T)$, where d is the thickness of the film and $\xi(T)$ is the coherence length. For $d/\xi(T)$ small, Ψ is constant across the film. At a critical value of $d/\xi(T)$, however, it becomes possible, qualitatively speaking, for a vortex line to fit into the film, with the result that Ψ changes phase across the film. This 'flux entry' effect is clearly seen in tunnelling data. A full theoretical and experimental study of type-II films in a parallel field was carried out by Guyon *et al* (1967).

With improvements in material preparation techniques, one can envisage superconducting cylinders with radius R comparable with $\xi(T)$, either free standing or as part of a composite. Like the Schrödinger equation for an electron in a wire in the presence of a parallel field (Landau and Lifshitz 1985, Constantinou *et al* 1992), the linearized GL equation for a cylinder in a parallel field can be solved exactly in terms of the confluent hypergeometric function. We present this solution here and apply it to the calculation of the critical field for a free-standing cylinder.

In the usual notation (Tilley and Tilley 1990) the linearized GL equation for Ψ is

$$(1/2\mu)(-i\hbar\nabla - 2eA)^2\Psi + \alpha\Psi = 0 \tag{1}$$

where μ is the electronic mass and the temperature enters via α:

$$\alpha = \alpha_0(T - T_c) \tag{2}$$

where α_0 is constant and T_c is the critical temperature. We apply this to a cylinder of radius R with magnetic field B along the axis. The vector potential can be taken as

$$A_r = A_z = 0 \qquad A_\phi = \tfrac{1}{2}Br. \tag{3}$$

The azimuthal dependence of Ψ is simple:

$$\Psi = \chi(r)\exp(\mathrm{i}m\phi) \tag{4}$$

and, as for the electron problem (Constantinou *et al* 1992), the equation for the radial part reduces to

$$\zeta^2\,\mathrm{d}^2\chi/\mathrm{d}\zeta^2 + \zeta\mathrm{d}\chi/\mathrm{d}\zeta + [|\alpha|\zeta/\hbar\omega_c - \tfrac{1}{4}(\zeta + m)^2]\chi = 0 \tag{5}$$

where ω_c is the cyclotron resonance frequency for a particle of mass μ and charge $2e$,

$$\omega_c = 2eB/\mu \tag{6}$$

and ζ is defined by

$$\zeta = r^2/2a_c^2 \tag{7}$$

where

$$a_c^2 = \hbar/2eB \tag{8}$$

defines the magnetic length a_c for a particle of charge $2e$. Equation (5) is brought into canonical form by the substitution (Landau and Lifshitz 1985)

$$\chi = \exp(-\zeta/2)\zeta^{|m|/2}W(\zeta). \tag{9}$$

The equation for W is Kummer's equation for the confluent hypergeometric function (Abramowitz and Stegun 1965):

$$\zeta\,\mathrm{d}^2W/\mathrm{d}\zeta^2 + (b - \zeta)\,\mathrm{d}W/\mathrm{d}\zeta - aW = 0 \tag{10}$$

where

$$a = \tfrac{1}{2} - |\alpha|/\hbar\omega_c + \tfrac{1}{2}|m| + \tfrac{1}{2}m \tag{11}$$

and

$$b = |m| + 1. \tag{12}$$

The solution of (10) is written in standard notation as

$$W(\zeta) = M(a, b, \zeta). \tag{13}$$

The second solution $U(a, b, \zeta)$ is discarded since it is divergent at $\zeta = 0$.

The boundary condition at the outer radius $r = R$ of the cylinder is $\mathrm{d}\Psi/\mathrm{d}r = 0$, which leads to

$$\tfrac{1}{2}(|m|/2f - 1)M(a, b, 2f) + M'(a, b, 2f) = 0 \tag{14}$$

and

$$f = R^2/4a_c^2 \tag{15}$$

is the dimensionless parameter used by Guyon *et al* (1967).

Equation (14) is the eigenvalue equation to determine the critical field B_c. As in the determination of H_{c2} and H_{c3}, for example, the 'eigenvalue' $|\alpha|$ is effectively the temperature. The interpretation of (14) is therefore as follows. We regard $|\alpha|$ as fixed; then for a given m equations (11), (12) and (14) are solved for B. The maximum B (as a function of m) is the critical field B_c.

It may be noted that this problem involves two length scales, the magnetic length a_c defined in (8) and the coherence length $\xi(T)$ defined by

$$\xi^2 = \hbar^2/2\mu|\alpha| = \hbar^2/[2\mu\alpha_0(T_c - T)]. \tag{16}$$

Likewise there are two energy scales $|\alpha|$ and $\hbar\omega_c$. The ratios satisfy

$$2|\alpha|/\hbar\omega_c = a_c^2/\xi^2. \tag{17}$$

Following Guyon *et al* (1967), we reduce the problem to dimensionless form by defining temperature and field parameters ϵ and f, where

$$\epsilon = R^2/4\xi^2 = \mu|\alpha|R^2/2\hbar^2 \tag{18}$$

and f is given by (15), which is also written as

$$f = BR^2\pi/2\Phi_0 \tag{19}$$

where $\Phi_0 = h/2e$ is the flux quantum. The solution may be represented as a universal curve of ϵ versus f.

The equations defining the numerical problem are the eigenvalue equation, equation (14), the definition of b, equation (12), and the definition of a, equation (11). In terms of ϵ and f, the latter is

$$a = \tfrac{1}{2} - \tfrac{1}{2}\epsilon/f + \tfrac{1}{2}|m| + \tfrac{1}{2}m. \tag{20}$$

Since f appears in both (14) and (20) whereas ϵ appears only in (20), a convenient numerical strategy is as follows. First fix f, then find $\epsilon(m)$; the minimum of $\epsilon(m)$ is the required point on the critical curve. Now b depends only on $|m|$, so (14) determines the same value of a for positive and negative m. It then follows from the definition of a, equation (11), that

$$\epsilon(m) = \epsilon(-m) + 2mf \qquad m = 0, \pm1, \pm2, \ldots. \tag{21}$$

Only the minimum value of ϵ has physical significance, so (21) shows that only negative values of m need be considered.

It is helpful to start with the limiting solutions for large and small R. For large R the curvature of the cylinder is unimportant and the ordinary surface-sheath result $H_{c3} = 1.69\ H_{c2}$ is recovered; in the present units this is

$$\epsilon = 0.592f \qquad \text{for } \epsilon \gg 1. \tag{22}$$

For small R, the solution must correspond to $m = 0$, since for $m \neq 0$ the kinetic-energy-type contribution to the Landau energy diverges as $R \to 0$. Since the small-ζ behaviour of the confluent hypergeometric function is $m(a, 1, \zeta) \sim$ constant as $\zeta \to 0$, it follows from (9) that the radial part $\chi(r)$ is almost constant. Equation (5) can then be integrated across the cylinder; the limits are $\zeta = 0$ corresponding to $r = 0$ and $\zeta = 2f$ corresponding to $r = R$. Thus

$$\int_0^{2f} \frac{|\alpha|\zeta}{\hbar\omega_c}\, d\zeta = \frac{1}{4}\int_0^{2f} \zeta^2\, d\zeta \tag{23}$$

which leads to

$$\epsilon = \tfrac{2}{3}f^2 \tag{24}$$

compared with $\epsilon = f^2/3$ for a film in a parallel field.

We have solved (14) numerically using standard software and our own subroutine for M. The resulting graph of ϵ versus f is shown in figure 1. For small f, $m = 0$ and (24) applies. There is then a succession of flux entry points at each of which m decreases by 1. The definitions of the reduced variables ϵ and f mean that the vertical scale is effectively temperature and the horizontal scale is magnetic field. The radius R scales both axes, so the value of the radius determines which section of the ϵ–f plot is experimentally accessible. An order of magnitude for values of R follows from (18): $\epsilon = 1$ corresponds to $R = 2\xi(T)$, where $\xi(T)$ is the usual coherence length. The present calculation is relevant to type-II superconductors, in which $\xi(T)$ is relatively small, so in practice a value of R in the range 10 to 50 nm might be needed for the present cylinder nucleation fields to be distinguished from the ordinary H_{c3} of a flat surface.

The nature of the state just below the critical field follows from (9), (13) and the small-ζ behaviour

$$M(a, b, \zeta) \sim 1 \qquad \text{for small } \zeta. \tag{25}$$

This gives

$$\chi \sim r^{|m|} \qquad \text{for small } r \tag{26}$$

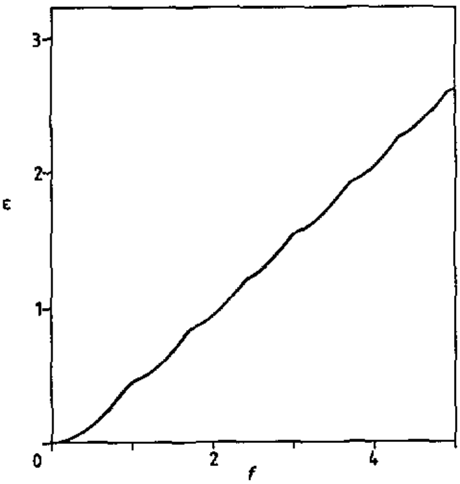

Figure 1. Graph of $\epsilon = \mu|\alpha|R^2/2\hbar^2$ versus $f = \pi BR^2/2\Phi_0$. The azimuthal quantum number m (fluxoid number) is 0 for small f and decreases by 1 at each flux entry point (derivative discontinuity). The given curve encompasses values of m between 0 and −7.

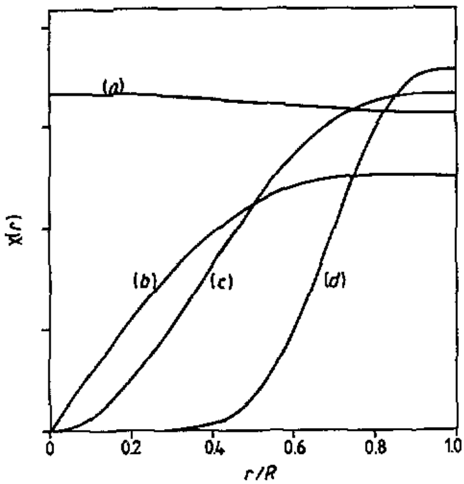

Figure 2. Radial dependence of the radial part of the wave function, $\chi(r)$, for (a) $m = 0$, $f = 0.5$; (b) $m = -1$, $f = 1.5$; (c) $m = -2$, $f = 2.0$; (d) $m = -10$, $f = 7.0$. The normalization of $\chi(r)$ is arbitrary.

so, even for $|m| = 1$, χ is rather small at the centre of the cylinder. This is confirmed by figure 2 showing the radial dependence of χ for a few values of m. It may be mentioned that the normalization of χ is not determined by the present calculation; further analysis including the non-linear term in the GL equation would be needed for this. It is helpful also to consider the expression for the supercurrent:

$$J = -(\mathrm{i}e\hbar/\mu)(\Psi^* \nabla\Psi - \Psi \nabla\Psi^*) - (4e^2/\mu)|\Psi|^2 A. \tag{27}$$

It follows from (3) and (4) and the fact that $\chi(r)$ is real that only the azimuthal component J_ϕ is non-zero; substitution of (3) and (4) in (27) gives

$$J_\phi = -(2e\chi^2/\mu)(\hbar|m|/r + eBr). \tag{28}$$

Because m is negative, the two terms in (27) have the same sign.

Equation (28) confirms the pciture given by figure 2. For small radius (on the scale of the coherence length) the superconducting phase nucleates uniformly across the cylinder. As $R/\xi(T)$ increases, the critical field corresponds to increasing (negative) values of m. The order parameter $\chi(r)$ and the current J_ϕ are increasingly conentrated at the outer radius of the cylinder with the interior region effectively in the normal phase, $\chi \simeq 0$.

We conclude with a brief review and comments on possible extensions. The physical results are summarized in figure 1. For a cylinder of given radius R the vertical axis ϵ is proportional to $T_c - T$; decreasing T corresponds to increasing the coherence length $\xi(T)$ and increasing $\epsilon = R^2/4\xi^2$. As ϵ increases, the critical field corresponds to increasing values of the fluxoid quantum number $|m|$. For a moderately large value of $|m|$ the superconducting state just below the critical field is described as a supercurrent of thickness about $\xi(T)$ around the perimeter of the cylinder, with the interior region normal. This is hardly surprising since the field is larger than the critical value H_{c2} for superconductivity to appear in the bulk.

For the quantum-mechanical problem, Makar *et al* (1991) give a discussion of a hollow cylinder. This could also be done for the present problem, the boundary condition $\mathrm{d}\Psi/\mathrm{d}r = 0$ being applied at the inner radius R_0 as well as the outer radius R. The solution for Ψ would be a linear combination $pM(a, b, z) + qU(a, b, z)$.

For a cylinder in a non-superconducting metallic matrix the boundary condition is modified; to be precise the condition of zero gradient is replaced by

$$\mathrm{d}\Psi/\mathrm{d}r - (1/\delta)\Psi = 0 \tag{29}$$

where δ is an extrapolation length (de Gennes 1966). This would lead to a modification of (14), but the calculation of the critical field is tractable.

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